Branch and Price for Job-Shop Scheduling with Time-Dependent Costs and Resource Constraints

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ISCO 2026, Kuşadası, Türkiye

Scheduling under Time-Varying Costs and Limits

- **Scheduling** problems: **manufacturing**, workforce management, computing.
- The **cost** of executing a task often depends on **when** it runs:
  - ▷ electricity tariffs,
  - ▷ compute pricing.
- Optimizing time-based metrics alone may lead to **high operating costs**.
- Resources consumed over time may also be **limited**:
  - ▷ contractual capacity agreements,
  - ▷ shared power budgets.

Problem Definition

- **Machines** $m \in \mathcal{M}$.
- **Jobs** $j \in \mathcal{J}$ to execute over a discrete time horizon $\mathcal{T}$:
  - » Processed over an ordered subset of machines, each exactly once.
- **Operations** $(j, m) \in \mathcal{J} \times \mathcal{M}$: processing of job j on machine m.
  - » Deterministic processing times $q_{j,m}$.

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- **Cardinality resource limit** $\overline{M}_t$: maximum simultaneously active machines at t.
- **Cost** $g^t_{j,m} \in \mathbb{R}_+$: incurred from executing (j, m) at $t \in \mathcal{T}$.

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Goal

Min **Total Operating Cost** (TOC)
s.t. operation **non-preemption**, **precedence**, machine **disjunction**, **cardinality** limit.

Energy-aware scheduling: instance and solution

Job j	1	2	3
Sequence M_j	$\{1,3,2\}$	$\{2,1,3\}$	$\{2,3,1\}$

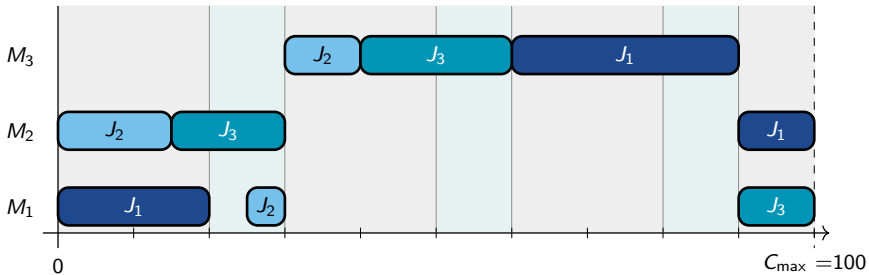
Machine m	1	2	3
duration $q_{1,m}$	20	10	30
duration $q_{2,m}$	5	15	10
duration $q_{3,m}$	10	15	20

Cost interval i	on-peak	off-peak
Tariff c^i	0.159	0.13
Length l^i	20	10

Machine m	1	2	3
Power φ_m	5	6	8

$g^t_{j,m} = \varphi_m \times c^i$, for t in interval i

Makespan minimization s.t. $\overline{M} = 2$ (TOC $= 204.7$)



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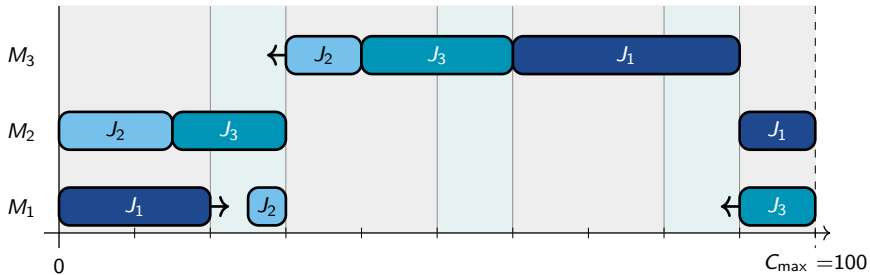

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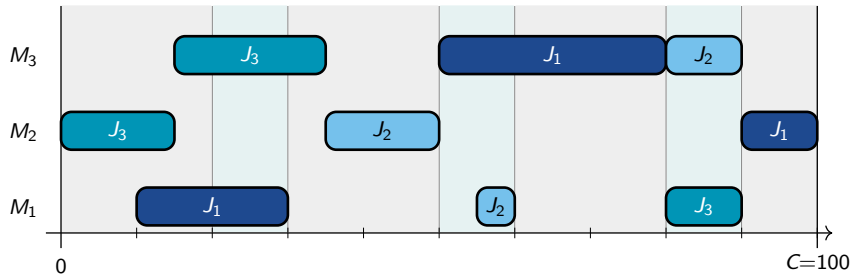
Operating Cost minimization s.t. $\overline{M} = 2$ and $C_{\max}$ (TOC = 197.1)



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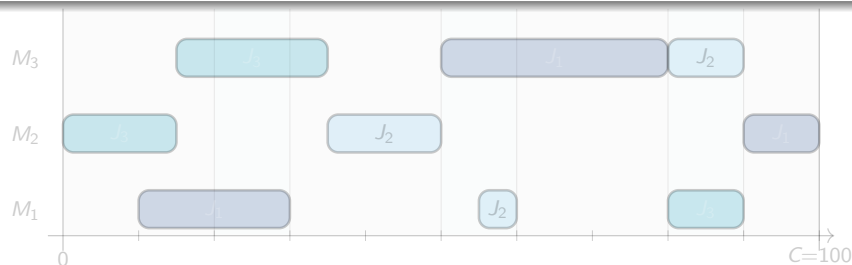
Can we exploit the cost structure?

Piecewise-constant profiles $\Rightarrow$ **Period-Indexed Formulations**: variables indexed on cost intervals.



Short literature review: energy-aware scheduling

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job-shop scheduling	Felloussi et al. [2026]	$Jm\|P_{max}\|$TOC	MILP (PI), B&C
	Felloussi et al. [2025]	$Jm\|\|$TOC	MILP (PI)
	Bley and Linß [2022]	$Jm\|$ on/off, r_j, $d_j\|$TOC	MILP (TI), B&B
	Masmoudi et al. [2019]	$Jm\|P_{max}\|$TOC	MILP (TI), MH
flexible job-shop scheduling	Park and Ham [2022]	$FJm\|$ on/off $\|C_{max},$TOC	MILP (TI), CP
	Jiang and Wang [2020]	$FJm\|\|C_{max},$TOC	MILP (TI), H
flow-shop scheduling	Ho et al. [2022]	$F2\|prmu,$ on/off $\|$TOC	MILP (PI), B&C
parallel machine scheduling	Gaggero et al. [2023]	$Pm\|\|C_{max},$TOC	MILP (TI), H
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PI: Period-Indexed formulation TI: Time-Indexed formulation

Table: An overview of related works.

- In practice: #periods $\ll$ #time steps $\Rightarrow$ PI outperform TI [Felloussi et al., 2026].

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$\rightarrow$ What if the costs (or resource limits) vary at **unit-time resolution**?

- Intermittent renewables $\Rightarrow$ price volatility (15-min, Nord Pool [2025]),
- Shared power capacity & non-controllable loads $\Rightarrow$ resource limit variation.

Time-Indexed Formulations

- Binary variables indicate whether (j, m) is "processed" at step t.
- Two compact variants [Artigues, 2017]:
  - » **Disaggregated**: strong LP relaxation, large model.
  - » **Aggregated**: weaker LP relaxation, smaller model.
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Idea

Decompose into **job-schedules**: offload part of the combinatorial work to a fast oracle.

Job-schedules

- Full schedule is decomposed into $|\mathcal{J}|$ **job-schedules**.
- Job-schedule $p \in \mathcal{P} \iff$ operation assignment *s.t* precedence and non-preemption:
  - » binary profile $a^p_{m,t} \equiv$ process on m at t,
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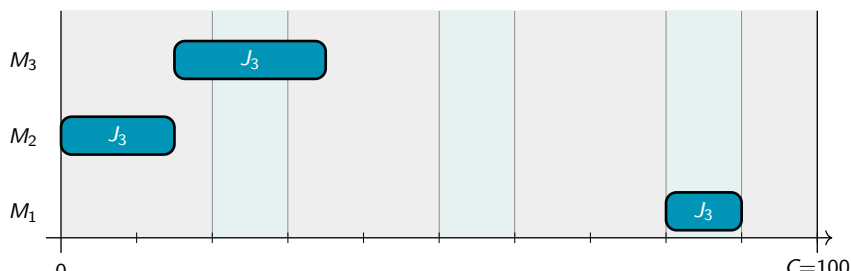

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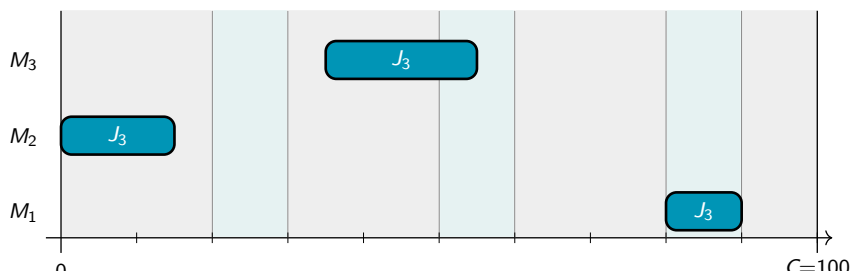

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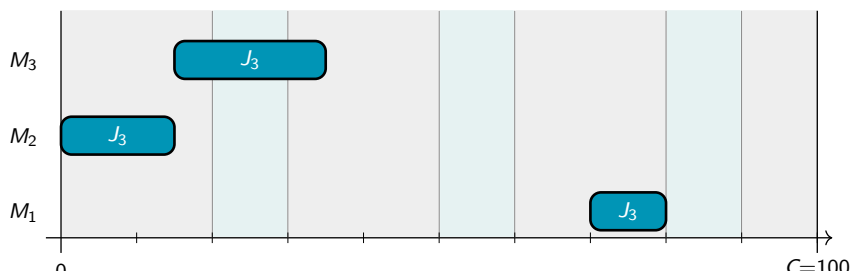
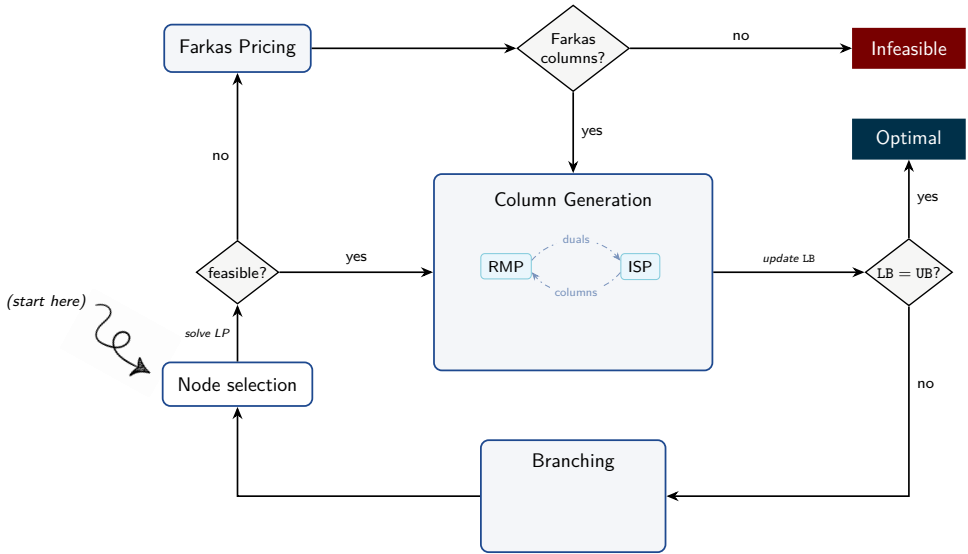


Extended Formulation

$$
\begin{aligned}
\textbf{(MP)} \quad \min \quad & \sum_{j\in\mathcal{J}}\sum_{p\in\mathcal{P}_j} c_p\lambda_p, & & \\
\text{s.t.} \quad & \sum_{p\in\mathcal{P}_j} \lambda_p = 1, & \forall j\in\mathcal{J}, & \quad \text{(job processing)} \\
& \sum_{j\in\mathcal{J}}\sum_{p\in\mathcal{P}_j} a^p_{m,t}\lambda_p \leqslant 1, & \forall m\in\mathcal{M}, \forall t\in\mathcal{T}, & \quad \text{(non-overlap)} \\
& \sum_{j\in\mathcal{J}}\sum_{p\in\mathcal{P}_j}\sum_{m\in\mathcal{M}} a^p_{m,t}\lambda_p \leqslant \overline{M}_t, & \forall t\in\mathcal{T}, & \quad \text{(cardinality)} \\
& \lambda^p \in \{0,1\} & \forall p\in\mathcal{P} := \bigcup_{j\in\mathcal{J}}\mathcal{P}_j. &
\end{aligned}
$$

- $|\mathcal{P}|$ is exponentially large $\Rightarrow$ manage via **column generation**.
- LP relaxation solved at each node of a branch-and-bound tree $\Rightarrow$ **Branch and Price**.

Algorithm Overview



Pricing Subproblem

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  $\Rightarrow$ The pricing problem is a **shortest-path** on a directed acyclic graph (DAG).
- Solved **efficiently** by a **forward dynamic program**.

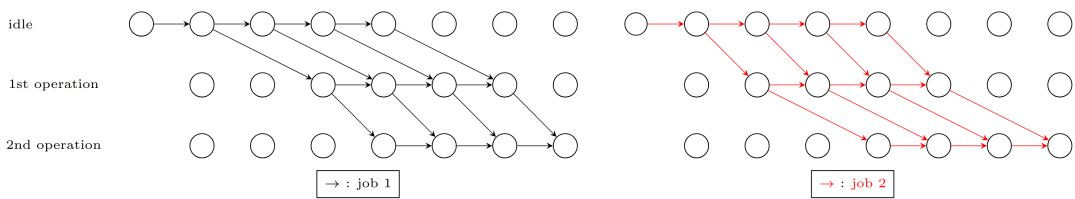

Figure: DAGs for pricing ($|\mathcal{J}| = |\mathcal{M}| = 2$)

LP Bound and Computational Advantage

- The induced path polytope is **integral**

  $\Rightarrow$ extended and disaggregated formulations share the **same LP bound**.

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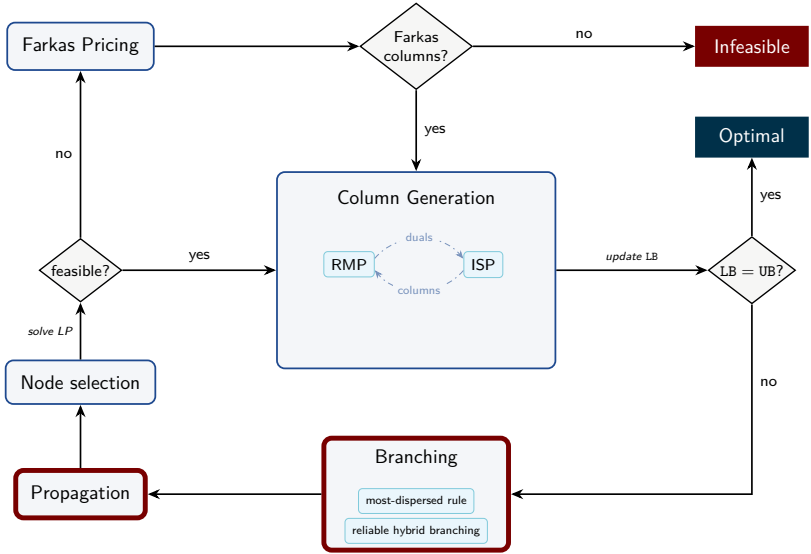
In this setting

Disaggregated relaxations are already tight, decomposition allows:

» Reduced explicit model size.

» Fast pricing oracle $\Rightarrow$ repeated calls are inexpensive.

Algorithm Overview



Branching

- **Branch** in the ISP on $z^t_{j,m} \in \{0,1\}$: whether (j,m) processes at t.
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- Variable selection: **hybrid reliable branching**
  - » most-dispersed rule: operation with most spread out execution in fractional solution,
  - » (quick) strong branching,
  - » pseudo-costs, complemented with problem-specific scores.

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- Number of induced fixings $\Rightarrow$ **inference score**, combined with pseudo-costs.
- Also invoked during strong branching.

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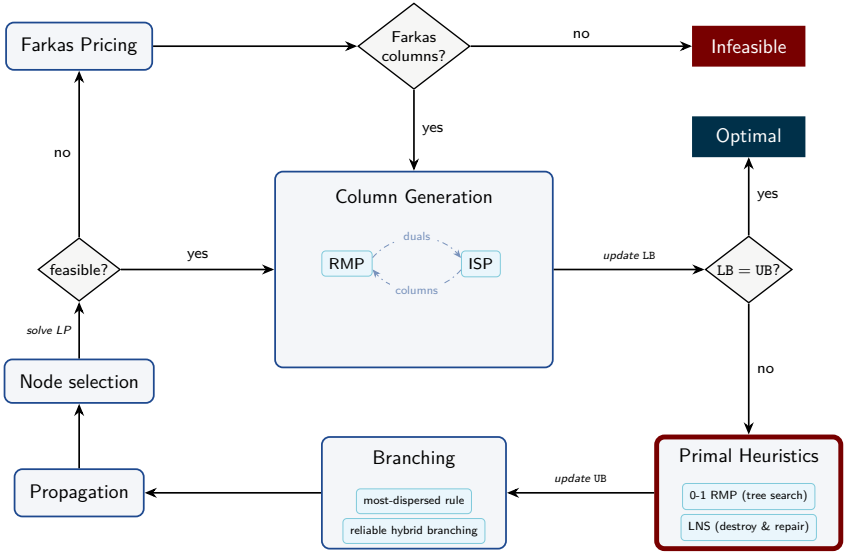

Primal Heuristics: Motivation

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  - » **Combine** columns from the **current pool** into primal solutions.
  - » But, are agnostic to the CG process: **heuristic limitations** or **integer infeasible**?

- Restricted master heuristic: solve $0-1$ (RMP) over current columns:
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Idea

- » **Enumerate** efficiently using the problem's structure.
- » **Explore** solution **neighborhoods** using the fast pricing DP.

Primal Heuristics: exact tree search

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On heuristic limitations

Tree-search optimality $\Rightarrow$ current best solution is found.

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Key Insight

Destroy-repair calls the **same DP pricing oracle** $\Rightarrow$ **inexpensive** cycles.

Additional components

- **Dual stabilization**
  - » exact pricing $\Rightarrow$ parameter-free dual **smoothing** [Pessoa et al., 2018],
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- **Early branching** without bound loss [Mehrotra and Trick, 1996]:
  - » exact pricing $\Rightarrow$ **Lagrangian lower bound** LB_{lag} [Lübbecke, 2011],
  - » **branch** before CG convergence if

$$\lceil z_{\text{RMP}} \rceil = \lceil LB_{\text{lag}} \rceil.$$

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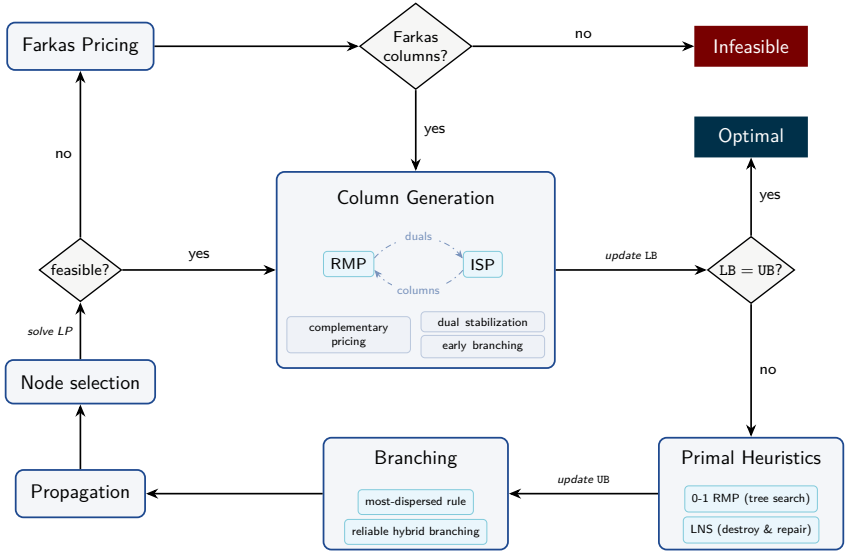

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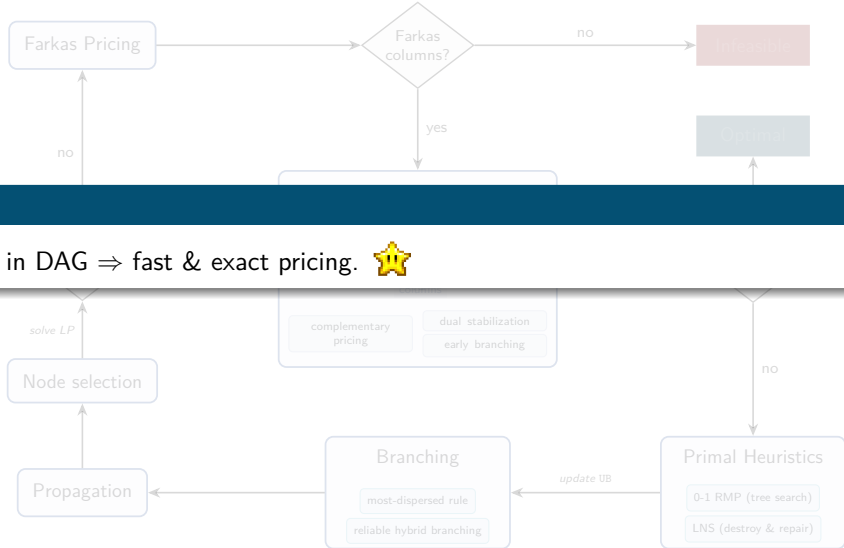

Key Insight

Shortest path in DAG ⇒ fast & exact pricing.

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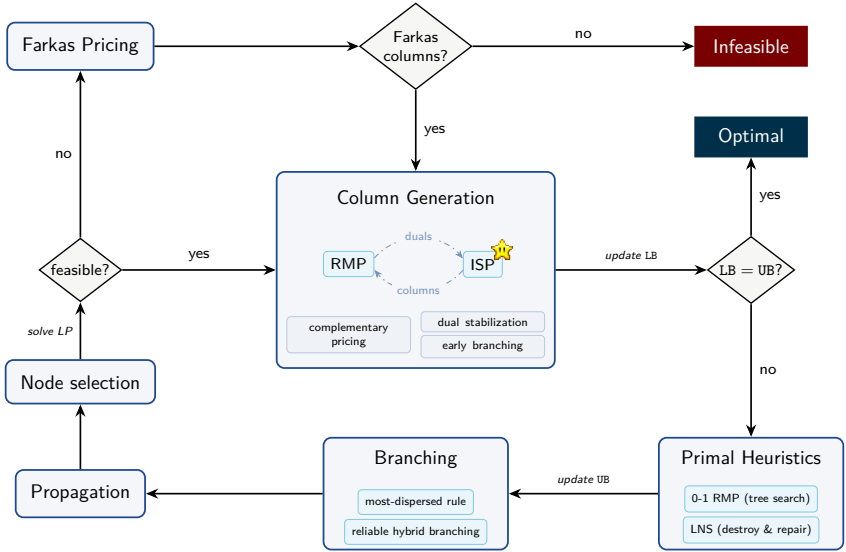

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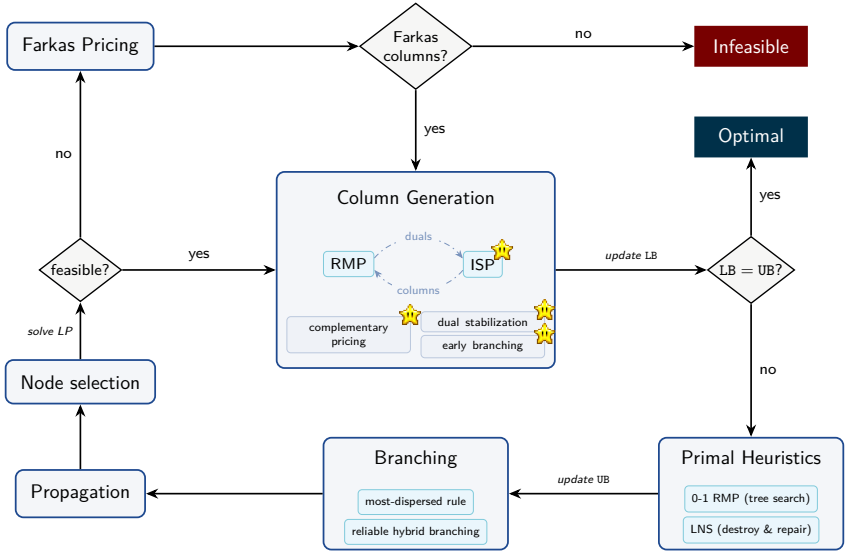

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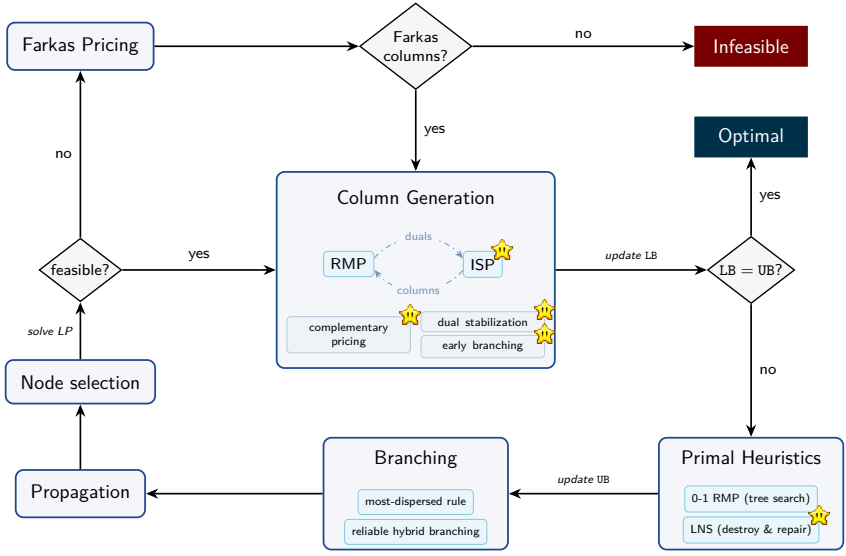

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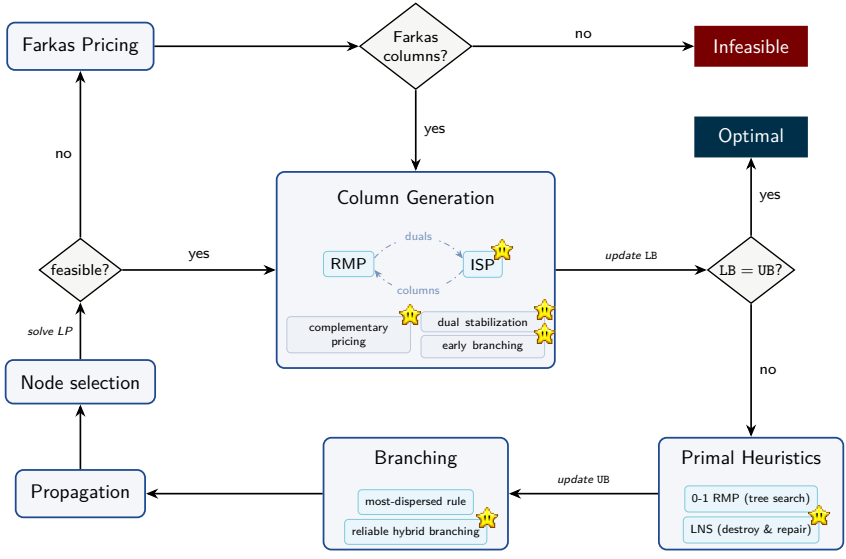

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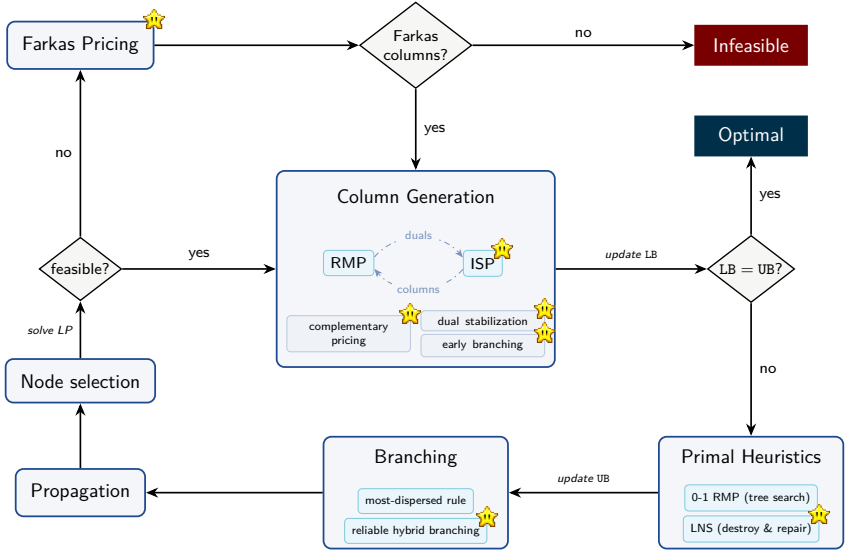

Outline

Instances and Setup

Instances generated following Felloussi et al. [2026] and Masmoudi et al. [2019]:

- 7 base JSSP instances.
- Time-dependent costs: 2 piecewise-constant profiles + 1 randomly drawn profile.
- Horizons $|\mathcal{T}| = \lceil \omega C^\star \rceil$:
  - $\omega \in \{1.2, 1.6, 2.0, 3.0\}$,
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B&P vs. Compact Formulations

A : Aggregated compact fomulation **D** : Disaggregated compact formulation **E** : Extended formulation

Instances		Branch and Bound (**A**)				Branch and Bound (**D**)				Branch and Price (**E**)			
subset	#inst.	#opt	#feas	T	#n	#opt	#feas	T	#n	#opt	#feas	T	#n
small	144	131	144	94.4	98.5	131	144	134.5	7.0	130	144	15.5	14.3
medium	144	24	99	2226.4	691.7	75	114	1327.3	3.1	110	133	41.2	13.6
large	216	0	54	TL	–	41	78	1813.1	3.0	150	187	79.4	15.3
all	504	155	297	154.4	133.4	247	336	415.9	4.8	**390**	**464**	**38.5**	14.5

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Outline

Ablation Study: Setup

- Tighter horizons $\omega \in \{1.2, 1.6\}$: 252 instances.
- Results on intersection subsets: **all-feas** and **all-opt**.
- Additional metrics:
  - **Primal** (PI) and **dual integral** (DI) [Berthold, 2013].
  - Time to **first solution** ($\%T^{1st}$) and time to **optimal primal bound** ($\%T^{opt}$).

Ablation Study: Primal Side

	all inst. (252)		all-feas (166)		all-opt (114)		
variant	#opt	#feas	PI	$\%T^{1st}$	$\%T^{opt}$	T	#n
basic	131	177	24	33	67	117.6	103
all-H	137	172	36	40	84	**72.3**	43
all	**138**	**212**	**17**	**21**	**86**	92.5	**36**

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- **Optimality: comparable performance** with and without heuristics.

Ablation Study: Dual Side

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variant	#opt	DI	T	#n
basic	131	3.3	113.8	102
all-Ps	127	**2.3**	147.2	87
all-Pr	**145**	2.6	102.6	52
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Table: Dual side features ablation comparison.

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- Propagation: **per-node overhead compounds** across the tree.
  - » Time: **+48%** to pricing and **+70%** to strong branching.

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→ all: **best** per-node **efficiency**, propagation is the remaining bottleneck.

Outline

Conclusion and outlook

- **Branch-and-price algorithm** for JSSP with time-dependent costs and cardinality resource constraints.
- **504 instances**: 77% solved to optimality, outperforming compact TI formulations.
- Fast pricing and tight original relaxation $\Rightarrow$ **Integrality property is not a limitation**.

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  - » Evaluation: benchmark on time-based objectives, other existing algorithms.
  - » Algorithm: RMP reoptimization is the bottleneck $\Rightarrow$ DOIs, cuts, stronger heuristics.

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marouane-f.github.io

marouane-f/branchprice_jssp

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Appendix: Dynamic Programming Recurrence

Let $d_j^\star(i,t)$ = minimum cost of any partial schedule completing operation i at time t:

$$d_j^\star(i,t) = \min\Big\{\underbrace{d_j^\star(i,t-1)}_{\text{idle}},\quad \underbrace{d_j^\star(i-1,\ t-q_i)+S_i^t}_{\text{processing}}\Big\}.$$

- Minimum reduced cost of any complete j-schedule: $d_j^\star(|\mathcal{M}|,|\mathcal{T}|)$.

Appendix: Pricing improvements

- **Dual stabilization**: parameter-free scheme of Pessoa et al. [2018] based on sub-gradient information $\Rightarrow$ reduces dual oscillations and speeds convergence.
- For any dual optimal solution of **(MP)**: $\alpha_j \geqslant \min_{p \in \mathcal{P}_j} c_p \Rightarrow$ enforced in the dual space as a variable in the primal.
- **Complementary pricing**: at each root-node CG iteration, an additional randomized pricing call forbids (j, m, t) with probability $\sum_{p \in \mathcal{P}_j} a^p_{m,t} \hat{\lambda}_p$, generating columns complementary to the active ones.
- **Window sums** S^t_i updated in $O(1)$ per step: $S^t_i = S^{t-1}_i - \hat{c}^{t-q_i}_i + \hat{c}^t_i$.
- **Time windows** from precedences restrict admissible (i, t) states $\Rightarrow$ arc fixings.

Appendix: Lagrangian Bound and Exact Branching

- With **exact pricing**, compute a **Lagrangian lower bound** LB_{lag} on the value of **(MP)** Lübbecke [2011].
- Data scaled to obtain an integer objective $\Rightarrow \lceil LB_{\text{lag}} \rceil$ is a valid lower bound.
- **Early branching** Mehrotra and Trick [1996]: branch before CG convergence if

$$\lceil z_{\text{RMP}} \rceil = \lceil LB_{\text{lag}} \rceil,$$

  without losing relaxation strength.
- Saves LP re-optimization iterations at nodes where the bound is already tight.

Appendix: Branching in the ISP

- Branch on original binary variables:

$$z^t_{j,m} = \sum_{p \in \mathcal{P}_j} a^p_{m,t} \lambda_p \in \{0, 1\},$$

  indicating whether (j, m) is processing at time t.

- Integrality of $\mathbf{z} \iff$ integrality of $\boldsymbol{\lambda}$ (set-partitioning argument).
- **Two branches**:
  - » $z^t_{j,m} = 0$: forbid processing transition of (j, m) at t in DP.
  - » $z^t_{j,m} = 1$: force (j, m) at t, forbid all $t' \neq t \Rightarrow$ also reduces to arc fixings in DP.
- Branching on master variables (resource, disjunction) led to large trees in preliminary experiments.

Appendix: Hierarchical Branching Scheme

- **Step 1 — Most-dispersed rule**: select the operation (j, m) whose execution is most spread in the fractional solution:

$$(j, m) = \arg\max_{(j,m)} |F_{j,m}|, \quad F_{j,m} := \{t : 0 < z^{\star t}_{j,m} < 1\},$$

  then branch on $t = \arg\max_{t \in F_{j,m}} z^{\star t}_{j,m}$.

- **Step 2 — Strong branching** (near root): on the candidates from Step 1, solve the **(RMP)** for each candidate branch without re-pricing; pick the best dual bound improvement.

- **Step 3 — Pseudo-cost branching**: once enough observations are collected, use pseudo-costs (complemented with aggregated operation- and time-level scores) to select the branching variable.

- Falls back to most-dispersed rule if no reliable pseudo-costs are available.

Appendix: Propagation

Fixings derived from branching decisions $\mathcal{B}_n$:

- **Job/machine disjunction**: if $z^t_{j,m} = 1$, then $z^t_{j,m'} = 0$ for all $m' \neq m$, and $z^t_{j',m} = 0$ for all $j' \neq j$.
- **Precedence**: if $z^t_{j,m} = 1$, fix $z^{t'}_{j,m'} = 0$ for $(j, m') \prec (j, m)$, $t' > t$ (and symmetrically for successors).
- **Non-preemption (contiguity)**: if $z^{t_1}_{j,m} = z^{t_2}_{j,m} = 1$, then $z^t_{j,m} = 1$ for all $t_1 < t < t_2$.
- **Non-preemption (interruption)**: if $z^{t_1}_{j,m} = 0$, $z^{t_2}_{j,m} = 1$, fix $z^t_{j,m} = 0$ for $t < t_1$ (when $t_1 < t_2$).
- **Cardinality limit**: if $\overline{M}_t$ machines are already fixed active at t, fix remaining machines idle.

Propagation is also invoked during strong branching.

Appendix: Exact Tree Search

- $0-1$ RMP is large and highly constrained.
- **Depth-first search**: column assignment per job in a predefined order of traversal.
- A node is **pruned** by comparing incumbent value against
  - Cost of partial solution at a node, or
  - LP-free additive lower bound LB
    - » LB $\coloneqq$ partial cost $+ \sum$ costs of cheapest compatible columns among remaining jobs

Appendix: Large Neighborhood Search

- Infeasible $\Rightarrow$ repair (find solution)
  - » **Repair**: reoptimize unfixed jobs in partial solutions.
  - » Same oracle as pricing, with different objective and fixings.

- Feasible $\Rightarrow$ destroy-repair (improve solution)
  - » A **destroy** set allows to define a neighborhood around an incumbent solution.
  - » Run pricing oracle over the neighborhood.

Appendix: Root LP Statistics

- Over all 504 instances, within the time limit:
  - » $\mathbf{E}$: **499** root relaxations solved.
  - » $\mathbf{A}$: **391** root relaxations solved.
  - » $\mathbf{D}$: **338** root relaxations solved.
- Average LP gap (root bound vs. optimal), computed over available instances:
  - » $\mathbf{E}$: **0.01%** $\mathbf{D}$: **0.01%** $\mathbf{A}$: **0.03%**
- $\mathbf{E}$ and $\mathbf{D}$ share the same LP relaxation strength (integrality property).
- Key insight: $\mathbf{D}$ finds more feasible/optimal solutions than $\mathbf{A}$, but fails at scale:
  - » 54 of the 168 infeasible $\mathbf{D}$ runs terminate due to **memory limits**.
  - » 66 fail to solve the **root LP** within the time limit on medium instances.