# A Branch-and-Cut Algorithm For Energy-Aware Scheduling

Marouane Felloussi  Xavier Delorme  Paolo Gianessi

Mines Saint-Etienne, LIMOS UMR 6158 CNRS, Saint-Etienne, France

## 1. Context and Objectives

Scheduling problems arise across manufacturing, workforce management, and computing. In many such settings, the **cost** of executing a task depends on **when** it runs (e.g., under electricity tariffs), and its **resource** consumption over time may be **limited** by contractual capacity agreements or shared power budgets.

Optimizing classical time-based metrics alone (e.g. makespan) may then lead to **high operating costs** or **infeasible resource use**.

→ We address the Job-Shop Scheduling Problem under *Time-of-Use (ToU) pricing* and a *peak-power limit*, minimizing total energy cost. We propose an exact **branch-and-cut** algorithm based on a **period-indexed formulation**, in which the peak-power limit is enforced through **clique-forbidding inequalities**. These inequalities are derived from the polyhedral description of the associated knapsack polytope.

## 2. Job-Shop Scheduling under Time-of-Use Pricing

### Problem definition

Given sets of machines $\mathcal{M}$, jobs $\mathcal{J}$, and operations $\mathcal{O} = \mathcal{J} \times \mathcal{M}$. Each operation $(j, m)$ has processing time $q_{j,m}$. Machine $m$ has nominal power $\varphi_m$. The horizon $C$ is split into ToU periods $p \in \mathcal{P}$ starting at $t^p$, of length $t^{p+1} - t^p = l^p$, and price $\kappa^p$. Power demand at any instant must not exceed $\overline{\varphi}$.

The problem asks to minimize total energy cost (TEC) subject to:

- operation full execution, non-preemption, precedence,
- machine non-overlap,
- the peak-power limit.

### Instance and Solution

| Job $j$ | 1 | 2 | 3 |
|---|---|---|---|
| Sequence $M_j$ | $\{1,3,2\}$ | $\{2,1,3\}$ | $\{2,3,1\}$ |

| ToU period $p$ | on-peak | off-peak |
|---|---|---|
| Tariff $\kappa^p$ | 0.159 | 0.13 |
| Duration $l^p$ | 20 | 10 |

| Machine $m$ | 1 | 2 | 3 |
|---|---|---|---|
| duration $q_{1,m}$ | 20 | 10 | 30 |
| duration $q_{2,m}$ | 5 | 15 | 10 |
| duration $q_{3,m}$ | 10 | 15 | 20 |

| Machine $m$ | 1 | 2 | 3 |
|---|---|---|---|
| Power $\varphi_m$ | 5 | 6 | 8 |

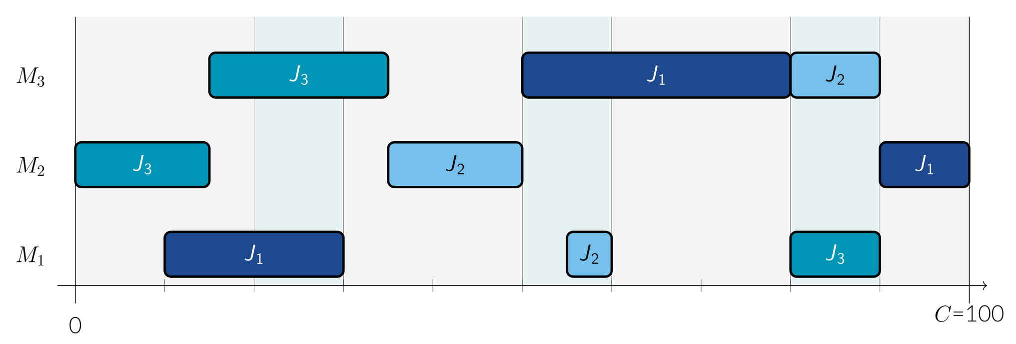



Figure 1. Energy cost minimization s.t. $\overline{\varphi} = 13 \Longrightarrow$ TEC = 197.1

## 3. A Period-Indexed Mixed-Integer Formulation

Variables:

▷ $x^p_{j,m} \in \{0,1\}$: operation activity over a period.

▷ $d^p_{j,m} \geqslant 0$: operation duration over a period.

▷ $s_{j,m} \geqslant 0$: operation starting date.

▷ $u^{j',m'}_{j,m} \in \{0,1\}$: precedence indicator.

Objective Function:

$$\min \sum_{p\in\mathcal{P}} \kappa^p \sum_{m\in\mathcal{M}} \varphi_m \sum_{j\in\mathcal{J}} d^p_{j,m}, \qquad (1)$$

Constraints:

$$\sum_{p\in\mathcal{P}} d^p_{j,m} = q_{j,m}, \qquad \forall (j,m) \in \mathcal{O}, \qquad (2)$$

$$d^p_{j,m} \leqslant l^p x^p_{j,m}, \qquad \forall (j,m) \in \mathcal{O}, \forall p \in \mathcal{P}, \qquad (3)$$

$$d^p_{j,m} \leqslant t^{p+1} - s_{j,m} + C(1 - x^p_{j,m}), \qquad \forall (j,m) \in \mathcal{O}, \forall p \in \mathcal{P}, \qquad (4)$$

$$d^p_{j,m} \leqslant c_{j,m} - t^p x^p_{j,m}, \qquad \forall (j,m) \in \mathcal{O}, \forall p \in \mathcal{P}, \qquad (5)$$

$$c_{j,m} \leqslant s_{j,m'}, \qquad \forall (j,m) \prec (j,m'), \qquad (6)$$

$$c_{j,m} - s_{j',m'} \leqslant C(1 - u^{j',m'}_{j,m}), \qquad \forall (j,m), (j',m') \in \mathcal{O}, \qquad (7)$$

$$u^{j',m}_{j,m} + u^{j,m}_{j',m} = 1, \qquad \forall m \in \mathcal{M}, \forall j, j' \in \mathcal{J}, \qquad (8)$$

`PeakPowerLimit`. (9)

**Why period-indexed?** Variables are indexed over periods of ToU profiles, which are piecewise constant. This yields a formulation with **substantially fewer variables** than discrete formulations [2].

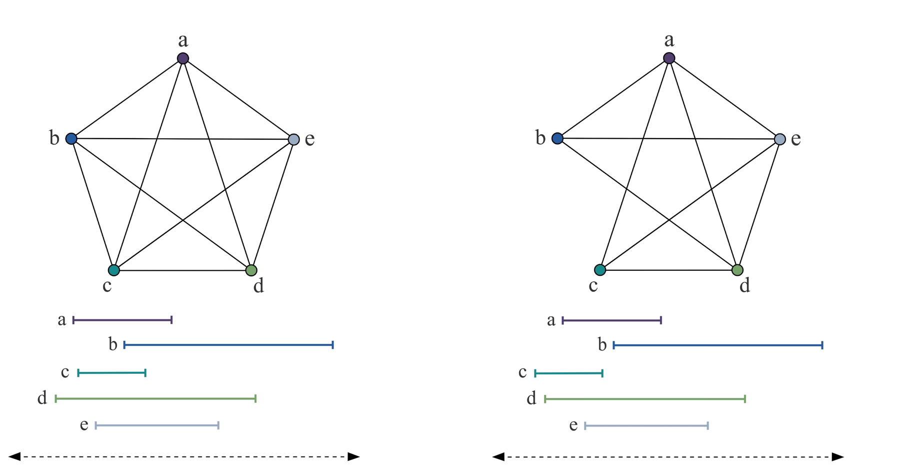
## 4. Peak-Power Limit by Forbidding Cliques

Let $z^{j',m'}_{j,m} := 1 - u^{j',m'}_{j,m} - u^{j,m}_{j',m'}$ be the overlap indicator. $z$ induces an **overlap graph** $\mathcal{G}(z)$: vertices are operations, edges connect overlapping pairs. A clique $K$ corresponds to operations that may run simultaneously, with weight $\varphi(K) = \sum_{v\in K} \varphi_v$.

If $\varphi(K) > \overline{\varphi}$, `PeakPowerLimit` is enforced by forbidding the clique:

$$\sum_{e\in E(K)} z^e \leqslant |E(K)| - 1. \qquad (9)$$

**Example.** $\varphi = [1, 3, 3, 4, 5]$, $\overline{\varphi} = 15$.




## 5. Forbidding Cliques: Minimal Covers are Sufficient

Consider the **knapsack set** $\mathbf{KP}^{\varphi,\overline{\varphi}} := \{x \in \{0,1\}^n : \sum_i \varphi_i x_i \leqslant \overline{\varphi}\}$, with $n := |\mathcal{M}|$.

A **minimal cover** $C$ of $\mathbf{KP}^{\varphi,\overline{\varphi}}$ satisfies:

- $\sum_{i\in C} \varphi_i > \overline{\varphi}$,
- $\sum_{i\in C\setminus\{j\}} \varphi_i \leqslant \overline{\varphi}$ for every $j \in C$.

Each minimal cover induces a clique-forbidding inequality, and it is **sufficient** to restrict to **cliques over minimal covers**.

The number of minimal covers is small, but the number of induced inequalities grows exponentially in $|\mathcal{J}|$ (number of $|C|$-permutations).

⇒ Embedded in a **branch-and-cut** framework: separate violated cuts only when an integer candidate is found.

## 6. How to Forbid Cliques over Extended Covers?

**Minimal cover** $C$: at most $|C| - 1$ machines among simultaneously active $|C|$-permutations ⇒ removing one edge suffices.

**Extended cover of** $C$: at most $r$ among $|C|$, with $r < |C| - 1$ ⇒ how many edges to remove?

At most $r$ active among $|C|$ ⟺ induced subgraph is $K_{r+1}$-free:

- The maximum number of edges in such a graph is bounded by the **Turán number** $\mathtt{ex}(n, K_r) = (1 - \frac{1}{r})\frac{n^2}{2}$.
- $\mathtt{ex}(n, K_r)$ can be used as the right-hand side of clique-forbidding inequalities.
- The graph induced by $z$ is an interval graph, which admits a tighter bound.

**Theorem.** *The maximum number of edges in an $n$-vertex $K_{r+1}$-free interval graph is*

$$\binom{n}{2} - \binom{n-r+1}{2}.$$

**Proof idea.** sort intervals by right endpoint and reason on the complement graph. (This rediscovers a similar result by [1].)

**Example.** $\varphi = [1, 3, 3, 4, 5]$ and $\overline{\varphi} = 10$.

▷ Minimal cover $\{3, 4, 5\}$ ($r = 2$, $n = 3$):

$$\Rightarrow z^4_3 + z^5_3 + z^5_4 \leqslant 2.$$

▷ Extended cover $\{2, 3, 4, 5\} \cup \{1\}$ ($r = 3$, $n = 5$):

$$\Rightarrow z^2_1 + z^3_1 + z^4_1 + z^5_1 + z^3_2 + z^4_2 + z^5_2 + z^4_3 + z^5_3 + z^5_4 \leqslant 7.$$

## 7. Extension to Arbitrary Inequalities

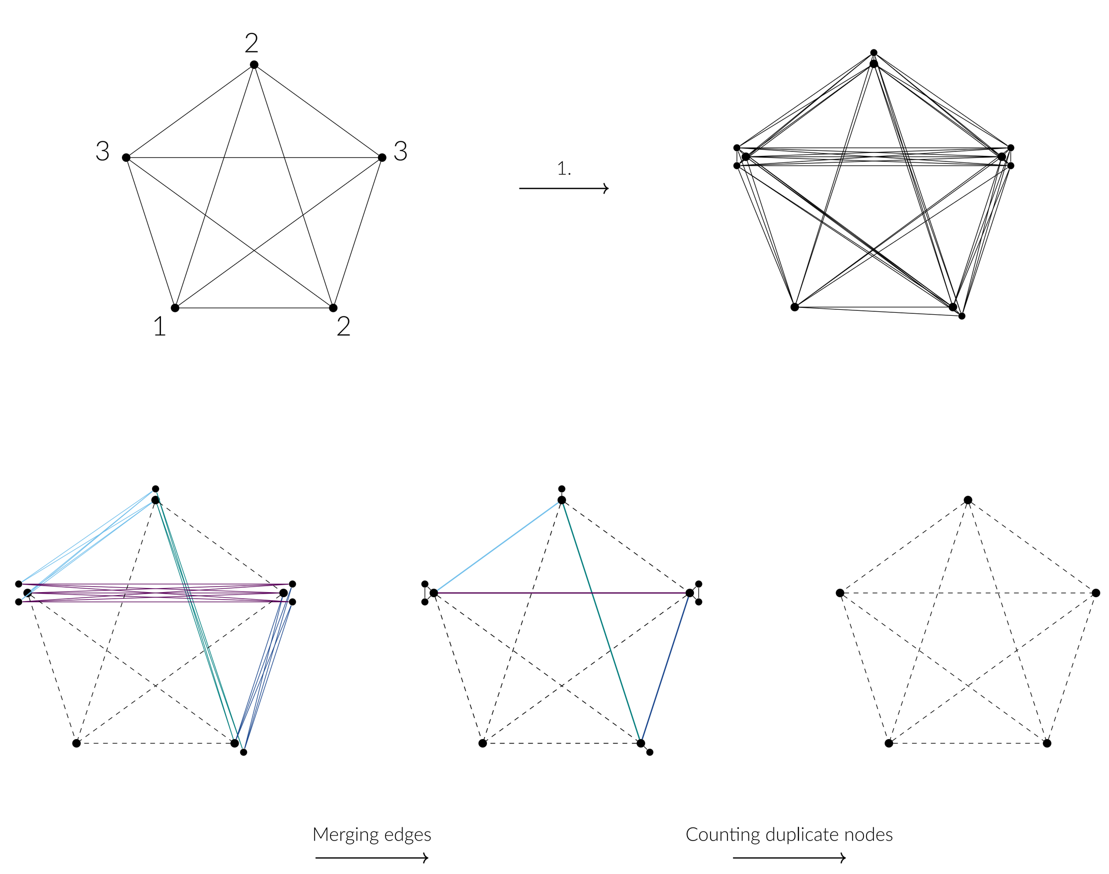
For any inequality $a^\top x \leqslant b$ valid for $\mathbf{conv}\{\mathbf{KP}^{\varphi,\overline{\varphi}}\}$, we derive a clique-forbidding cut via an **interval-replication argument** on the weighted graph $\mathcal{G}_a = (\mathcal{V}_a, \mathcal{E}_a)$:

If $a \in \{0,1\}^n$: minimal and extended covers. Otherwise:

1. replace each vertex $v$ by $a_v$ identical copies (pairwise overlapping), yielding a multiset $\mathcal{V}^1_a$ with $|\mathcal{V}^1_a| = \sum_{v\in\mathcal{V}_a} a_v$,
2. apply the interval-graph Turán bound on the resulting unit-weight graph,
3. merge redundant edges and account for duplicate vertices.

This yields the unified formula:

$$\sum_{(u,v)\in\mathcal{E}_a} a_u a_v z^v_u \leqslant \binom{|\mathcal{V}^1_a|}{2} - \binom{|\mathcal{V}^1_a| - b + 1}{2} - \sum_{u:a_u \geqslant 2} \binom{a_u}{2}.$$




## 8. Valid Inequalities

Additional inequalities are valid for the formulation and strengthen its linear relaxations, while keeping the model compact.

### Capacity-related inequalities

Let $D^p_m := \sum_{j\in\mathcal{J}} d^p_{j,m}$ and $X^p_m \in \{0,1\}$ with $X^p_m \geqslant x^p_{j,m}, \forall j \in \mathcal{J}$.

- Period packing: $\sum_{m\in\mathcal{M}} a_m D^p_m \leqslant b \times l^p$, with $p \in \mathcal{P}$, and $a^\intercal \mathbf{x} \leqslant b$ a valid inequality for $\mathbf{conv}\{\mathbf{KP}^{\varphi,\overline{\varphi}}\}$.
- Period flow cover:
$$\sum_{m\in F} \varphi_m \frac{D^p_m}{l^p} + \sum_{m\in F^+} (\varphi_m - \lambda)^+ (1 - X^p_m) \leqslant \overline{\varphi},$$
with $p \in \mathcal{P}$, and $F$ a flow cover of $\mathbf{KP}^{\varphi,\overline{\varphi}}$.

### Scheduling-related inequalities

- Job non-overlap: $\sum_{j\in\mathcal{J}} d^p_{j,m} \leqslant l^p X^p_m, \quad \forall m \in \mathcal{M}, \forall p \in \mathcal{P}.$
- Machine non-overlap: $\sum_{m\in\mathcal{M}} d^p_{j,m} \leqslant l^p, \quad \forall j \in \mathcal{J}, \forall p \in \mathcal{P}.$
- Generalized non-preemption:
$$d^p_{j,m} \geqslant l^p (x^{p_1}_{j,m} + x^{p_2}_{j,m} - 1), \quad \forall (j,m) \in \mathcal{O}, \forall p_1 < p < p_2 \in \mathcal{P}.$$

## 9. Computational experiments

**Setup.** Implemented in `Julia` + `Gurobi`; `Polymake` for polyhedral descriptions. 1-hour time limit per instance, exclusive runs.

**Benchmark.** Existing 6×6 and 5×10 instances + new 6×8, 7×7, 6×10, 7×9, 8×8 instances with four knapsack-structured power profiles, multiple scheduling horizons and peak limits. [4].

**Results.**

**60/60**
literature instances solved to optimality
*avg. < 90s, vs. ∼2000s (∼25×) for the time-indexed reference [5]*

**72/78**
new instances solved to optimality
*extending the benchmark to 6×8, 7×7 and smaller power limits*

**−10% runtime, +1 instance**
adding all clique-forbidding cuts
*over the minimal-cover-only variant on a hard 72-instance benchmark*

## 10. Conclusion

**Exploit the structure of the cost.** Indexing variables on ToU periods, rather than discretizing the full horizon, yields a compact model that, suitably strengthened, outperforms time-indexed formulations.

**Valid inequalities matter.** Indeed, several families of valid inequalities are needed to make the period-indexed formulation competitive [3], without compromising the compactness of the model.

**Generic separation, principled extensions.** The clique-forbidding scheme separates an exponential family of constraints in polynomial time based on the minimal covers of the peak-power knapsack set. It extends to extended covers via a Turán-type bound for interval graphs, and generalizes to arbitrary inequalities through a graph-replication argument. This framework extends to other similarly constrained scheduling problems with modest adjustments.

## References

[1] Harvey Abbott and Meir Katchalski. A Turán type problem for interval graphs. *Discrete Mathematics*, 25(1):85–88, 1979.

[2] Christian Artigues. On the strength of time-indexed formulations for the resource-constrained project scheduling problem. *Operations Research Letters*, 2017.

[3] Marouane Felloussi, Xavier Delorme, and Paolo Gianessi. Minimizing energy cost in a job-shop scheduling problem under ToU pricing: a new method based on a period-indexed MILP. In *Proceedings of the 14th International Conference on Operations Research and Enterprise Systems - ICORES*. INSTICC, SciTePress, 2025.

[4] Marouane Felloussi, Xavier Delorme, and Paolo Gianessi. A branch-and-cut algorithm for job-shop scheduling under time-of-use pricing and a peak power limit. *European Journal of Operational Research*, 2026.

[5] Oussama Masmoudi, Xavier Delorme, and Paolo Gianessi. Job-shop scheduling problem with energy consideration. *International Journal of Production Economics*, 216:12–22, 10 2019.

Full paper (*European Journal of Operational Research*)